University of Toronto
Scarborough Campus
October 28, 1998

CSC B38F Midterm Examination

Aids allowed: One 8.5×11 handwritten, non-photocopied 'cheat sheet'

Duration: One hour and fifty minutes

- There should be 9 pages in this exam booklet, including this cover page.
- Answer all questions.
- Put all answers in this booklet, in the spaces provided.
- For rough work, use the backs of the pages; ***these will not be marked***.
- Good luck!

Name ______________________________

Student Number ______________________________

Problem	Marks Rec'ved	Marks Worth
1.		10
2.		20
3.		20
4.		30
5.		20
TOTAL		100

QUESTION 1. (10 marks)

Use induction to prove that for every $n \in \mathbb{N}$, 7 divides $8^n - 1$.

PROOF:

QUESTION 2. (20 marks)

Consider the following recursive definitions of two functions from $\mathbb{N}$ to $\mathbb{N}$:

$$F(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F(n-1) + F(n-2), & \text{if } n > 1 \end{cases}$$

and

$$G(n) = \begin{cases} 1, & \text{if } n = 0 \\ 2, & \text{if } n = 1 \\ G(n-1) + G(n-2) + 1, & \text{if } n > 1 \end{cases}$$

Prove that, for any $n \in \mathbb{N}$, $G(n) = F(n+3) - 1$.

PROOF:

QUESTION 3. (20 marks)

For each assertion below, state whether it is true or false (by circling the corresponding word), and justify your answer in the space provided. ***No credit without correct justification.***

a. $n + \log_2 n \in O(n)$. **TRUE / FALSE**

b. $n - \sqrt{n} \in \Omega(n)$. **TRUE / FALSE**

c. $17n \in \Theta(n^2)$. **TRUE / FALSE**

d. For any function $f : \mathbb{N} \to \mathbb{N}$, if $f \in \Theta(n^2)$ then $f \in O(n^3)$. **TRUE / FALSE**

QUESTION 4. (20 marks)

Define $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ by the following recurrence. (Recall that $\mathbb{Z}^+$ is the set of positive integers.)

$$f(n) = \begin{cases} 6, & \text{if } n = 1 \\ 16, & \text{if } n = 2 \\ 4 \cdot f(n-2) + 10 \cdot 2^n, & \text{if } n \geq 3 \end{cases}$$

Prove that $f(n) \in O(n \cdot 2^n)$.

PROOF:

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QUESTION 5. (30 marks)

Consider the following program:

$x := 2 \cdot m$; $y := n$
while $x \neq 0$ **do**
  if $y \geq 1$ **then**
    $y := y - 3$; $x := x + 2$
  else
    $x := x - 2$
  end if
end while

Suppose that the following preconditions hold before the program is executed:

Preconditions: $m, n \in \mathbb{N}$.

a. (15 marks) Prove that if the preconditions hold before the program starts execution, the following predicate $P(k)$ is an invariant for the loop:

$P(k)$: if the k-th iteration of the loop is executed, then x_k is an even natural number and y_k is an integer such that $y_k \geq -2$

PROOF:

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b. (15 marks) Prove that, if the preconditions hold before the program starts, then the program will halt. (Hint: Consider the quantity $x + y + 2$ and use the fact that $P(k)$ is an invariant for the loop.)

PROOF:

THE END