University of Toronto
St. George Campus

Computer Science 238
Winter 2001

Midterm Exam
1 March, 2001

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Last Name:

First Name:

Student Number:

Rules:

1. Legibly write your name on this page and every page that you wish to be marked or returned. The pages of this exam will be separated during marking. *Unnamed pages will not be marked.*
2. There are 5 problems of equal weight, of which you must choose 4. Indicate below which problem you did NOT choose. *It will not be marked, regardless of whether work appears for it. If you do not indicate, it will be assumed that you did not choose problem 1.*
3. Total time is 50 minutes.
4. The exam is closed book, and no aids of any kind are allowed.
5. Write your answers directly on the exam in the space provided, or on the blank pages provided at the end. For each blank page that you use, write the number of the problem that you are solving on that page. Do not write the answer to more than one problem on the same blank page.
6. For rough work, you may use the back of any page. *These will not be marked.*
7. The instruction, "prove inductively," means that you must prove the given proposition by induction, complete induction, or structural induction. No credit will be given for (even very clever) non-inductive proofs.

Complete this line: I chose not to solve Problem __________.
Do not write below this line.

1:

2:

3:

4:

5:

Total:

Name:

1. (25 marks)

(a) (5 marks) State the general divide-and-conquer recurrence theorem.

(b) (5 marks) Find a tight asymptotic bound for the following recurrence:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1 \\ 3f(\lfloor \frac{n}{3} \rfloor) + 6n & \text{if } n > 1 \end{cases}$$

(c) (15 marks) Prove inductively that for all natural numbers $n \geq 2$, n has a prime factorisation.

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2. (25 marks) Let A be the smallest set such that:

1. $1 \in A$,
2. if $a \in A$, then $(-a) \in A$, and
3. if $a_1 \in A$ and $a_2 \in A$, then $(a_1 \times a_2) \in A$.

Let $\mathbf{sym} : A \longrightarrow \mathcal{N}$ be defined such that $\mathbf{sym}(a)$ is the total number of symbols (including parentheses) in a. Furthermore, let $\mathbf{eval} : A \longrightarrow \{-1, 1\}$ be the function that *arithmetically evaluates* expressions in A, i.e., if you type a into a calculator and press '=', you get the answer $\mathbf{eval}(a)$.

Prove inductively that, for all $a \in A$, $\mathbf{eval}(a) = (-1)^{\mathbf{sym}(a)+1}$.

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3. (25 marks) Consider the following proposition: if $f_1 \in \Theta(g_1)$ and $f_2 \in \Theta(g_2)$, then $(f_1 + f_2) \in \Theta(\max(g_1, g_2))$.

Is this true? If so, prove it. If not, provide a counter-example.

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4. (25 marks) Consider the following program (this is a recursive version of the program in Problem 5):

Precondition: A is a sorted array of integers, A[f ... l] is of odd length, $1 \le f \le l \le \text{length}(A)$.
Postcondition: Returns m such that $A[\frac{f+l}{2}] = m$.

```
median(f,l,A)
  if (f==l) then
    return A[f]
  else
    return median(f+1,l-1,A)
end
```

(a) (10 marks) State a recurrence for the worst-case time complexity of this algorithm.
(b) (15 marks) Prove inductively that the worst-case time complexity of this algorithm is tightly bounded at linear.
(**Hint**: choose your measure of size in (a) carefully!).

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5. (25 marks) Consider the following program and specification (this is an iterative version of the program in Problem 4):

Precondition: A is a sorted array of integers, $A[f \ldots l]$ is of odd length, $1 \leq f \leq l \leq \text{length}(A)$.
Postcondition: Returns m such that $A[\frac{f+l}{2}] = m$.

```
median(f,l,A)
  x := f; y := l;
  while (x != y) do
    x := x+1;
    y := y-1
  end while;
  return A[x]
end
```

(a) (10 marks) State (but do not prove) a loop invariant lemma that will allow you to solve (b).
(b) (15 marks) Prove that this algorithm is correct with respect to its specification.

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